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FAST TRACK COMMUNICATION

Statistical correlations in a Coulomb gas with a test charge

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Abstract
A recent paper (Jokela *et al* 2008 *Preprint* arXiv:0806.1491) contains a surmise about an expectation value in a Coulomb gas which interacts with an additional charge ξ that sits at a fixed position. Here, I demonstrate the validity of the surmised expression and extend it to a certain class of higher cumulants. The calculation is based on the analogy to statistical averages in the circular unitary ensemble of random-matrix theory and exploits properties of orthogonal polynomials on the unit circle.

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1. Purpose and result

In a recent paper Jokela, Järvinen and Keski-Vakkuri studied n-point functions in timelike boundary Liouville theory via the analogy to a Coulomb gas on a unit circle [1]. In this analogy, N unit charges at position t_i interact with additional charges of integer value ξ_a, situated at position τ_a. To illustrate this technique the authors of [1] considered the canonical expectation value

$$\langle\cdot\rangle \equiv \frac{1}{Z}\int\prod_{i=1}^{N}\frac{\mathrm{d}t_i}{2\pi}\prod_{i<j}|\mathrm{e}^{\mathrm{i}t_i}-\mathrm{e}^{\mathrm{i}t_j}|^2\prod_i|\mathrm{e}^{\mathrm{i}\tau}-\mathrm{e}^{\mathrm{i}t_i}|^{2\xi}(\cdot) \tag{1}$$

(where Z is a normalization factor so that $\langle 1\rangle = 1$) and surmised that

$$\langle \mathrm{Re}\, a_1\rangle \equiv \left\langle\sum_i\cos(\tau-t_i)\right\rangle = -\frac{\xi N}{N+\xi}. \tag{2}$$

In this communication I demonstrate the validity of (2) and also compute expectation values of the more general quantities

$$a_n \equiv \sum_{i_1<i_2<\cdots<i_n}\exp\left(\mathrm{i}\sum_{k=1}^{n}(t_{i_k}-\tau)\right). \tag{3}$$

As a result, I find

$$\langle a_n \rangle = (-1)^n \frac{(N-n+1)^{(\xi)}(n+1)^{(\xi-1)}}{(N+1)^{(\xi)}(1)^{(\xi-1)}} \quad \forall \quad n = 1, 2, \ldots, N, \qquad \xi \geqslant 0, \tag{4}$$

where $(x)^{(y)} = \Gamma(x+y)/\Gamma(x)$ is the generalized rising factorial (Pochhammer symbol). In particular, the validity of (2) follows from (4) by setting $n = 1$.

Expression (4) will be obtained by relating the generating polynomial

$$\varphi_{N,\xi}(\lambda) \equiv \sum_{n=0}^{N} \langle a_n \rangle (-\lambda)^{N-n} \tag{5}$$

to a weighted average of the secular polynomial in the circular unitary ensemble (CUE). This in turn establishes a relation to the Szegő polynomial of a Toeplitz matrix composed of binomial coefficients. This calculation sidesteps Jack polynomials and generalized Selberg integrals, which can be used to tackle general expectation values in multicomponent Coulomb gases [2].

2. Reformulation in terms of random matrices

The CUE is composed of $(N \times N)$-dimensional unitary matrices U distributed according to the Haar measure. Identify t_i with the eigenphases of such a matrix. The joint probability distribution is then given by [3]

$$P\left(\{t_i\}_{i=1}^N\right) = z \prod_{i<j} |\mathrm{e}^{\mathrm{i}t_i} - \mathrm{e}^{\mathrm{i}t_j}|^2, \tag{6}$$

where z is again a normalization constant. This expression can also be written as the product of two Vandermonde determinants $\det V^+ \det V^-$ with matrices $V^\sigma_{lm} = \mathrm{e}^{\mathrm{i}\sigma(m-1)t_l}$. Furthermore, we can write

$$\prod_i |\mathrm{e}^{\mathrm{i}\tau} - \mathrm{e}^{\mathrm{i}t_i}|^{2\xi} = [\det(1 - U\,\mathrm{e}^{-\mathrm{i}\tau})\det(1 - U^\dagger\,\mathrm{e}^{\mathrm{i}\tau})]^\xi. \tag{7}$$

Finally, the expressions a_n in (3) arise as the expansion coefficients of the secular polynomial

$$\det(U\,\mathrm{e}^{-\mathrm{i}\tau} - \lambda) = \sum_{n=0}^{N} a_n(-\lambda)^{N-n}. \tag{8}$$

Note that in all these expressions τ can be shifted to any fixed value by a uniform shift of all t_i's, which leaves the unitary ensemble invariant. Therefore, the expectation values are independent of τ. Collecting all results, we have the identity

$$\varphi_{N,\xi}(\lambda) = \frac{\langle[\det(1-U)\det(1-U^\dagger)]^\xi \det(U-\lambda)\rangle_{\mathrm{CUE}}}{\langle[\det(1-U)\det(1-U^\dagger)]^\xi\rangle_{\mathrm{CUE}}}. \tag{9}$$

This can be interpreted as a weighted average of the secular polynomial in the CUE.

3. Random-matrix average

Statistical properties of the secular polynomial without the weight factor ($\xi = 0$) have been considered in [4]. Clearly, $\varphi_{N,0} = (-\lambda)^N$, so that in this case the attention quickly moves on to higher moments of a_n. The main technical observation in [4] which allows us to address the case of finite ξ concerns averages of expressions $g\left(\{t_i\}_{i=1}^N\right)$ that are completely symmetric in all eigenphases. In this situation the average can be found via

$$\left\langle g\left(\{t_i\}_{i=1}^N\right)\right\rangle_{\mathrm{CUE}} = \int \prod_i \frac{\mathrm{d}t_i}{2\pi} g\left(\{t_l\}_{l=1}^N\right) \det W, \tag{10}$$

where $W_{lm} = \mathrm{e}^{\mathrm{i}t_m(l-m)}$. Equation (10) is simpler than the general expression involving the product of two Vandermonde matrices, since each eigenphase only appears in a single column of W.

In the present problem, the numerator in (9) is represented by the completely symmetric function

$$g_1\big(\{t_i\}_{i=1}^N\big) = \prod_{i=1}^N [(\mathrm{e}^{\mathrm{i}t_i} - \lambda)(1 - \mathrm{e}^{\mathrm{i}t_i})^\xi (1 - \mathrm{e}^{-\mathrm{i}t_i})^\xi], \tag{11}$$

while for the denominator we need to consider the similar expression

$$g_2\big(\{t_i\}_{i=1}^N\big) = \prod_{i=1}^N [(1 - \mathrm{e}^{\mathrm{i}t_i})^\xi (1 - \mathrm{e}^{-\mathrm{i}t_i})^\xi]. \tag{12}$$

Using the multilinearity of the determinant we can now pull each factor into the ith column and perform the integrals. This delivers the representation

$$\varphi_{N,\xi}(\lambda) = \frac{\det(B - \lambda A)}{\det A}, \tag{13}$$

where the matrices $A_{lm} = (-1)^{l-m}\binom{2\xi}{\xi+l-m}$, $B_{lm} = (-1)^{l-m+1}\binom{2\xi}{\xi+l-m+1}$ have entries given by binomial coefficients. We now exploit the regular structure of these matrices in two steps.

(i) Matrix B contains the same entries as matrix A, but shifted to the left by one column index. In order to exploit this, let us expand the determinant in the numerator into a sum of determinants of matrices labeled by $X = (x_m)_{m=1}^N$, where we select each column either from A ($x_m = \mathrm{A}$) or from B ($x_m = \mathrm{B}$). (Note that we set these symbols in roman letters.) The related structure of A and B then entails that $\det X$ vanishes if X contains a subsequence $(x_m, x_{m+1}) = (\mathrm{A}, \mathrm{B})$. Consequently we only need to consider determinants of matrices $X_n \equiv (\mathrm{B})_{m=1}^n \oplus (\mathrm{A})_{m=n+1}^N$, associated with sequences that contain n leading B's and $N-n$ trailing A's. As A is multiplied by $-\lambda$, $\det X_n$ contributes to order $(-\lambda)^{N-n}$. (Note that $X_0 = A$ and $X_N = B$.)

(ii) Next, consider the matrix A_{N+1}, where the subscript denotes the dimension, and strike out the first row and the $n+1$st column ($n = 0, 1, 2, \ldots, N$). This takes exactly the form of the matrix X_n of dimension N. Therefore, the expressions $(-1)^n \det X_n$ are the cofactors of the first row of A_{N+1}. These, in turn, are proportional to the first column of A_{N+1}^{-1}, where the proportionality factor is given by $\det A_{N+1}$. Consequently, taking care of all alternating signs,

$$\varphi_{N,\xi}(\lambda) = (-1)^N \frac{\det A_{N+1}}{\det A_N} \sum_{n=0}^N \big(A_{N+1}^{-1}\big)_{1,1+n} \lambda^{N-n}. \tag{14}$$

Via steps (i) and (ii) we have eliminated any reference to the matrix B.

4. Orthogonal polynomials

Matrix A is a Toeplitz matrix, $A_{lm} = c_{l-m}$. In order to find the explicit expression (4) we now make contact to the theory of orthogonal polynomials on the unit circle [5]. Among its many applications, this theory provides a general expression for the inverse of any Toeplitz matrix in terms of Szegő polynomials $\psi_N(\lambda)$. For the case of real symmetric coefficients, the inverse is generated via

$$\frac{\lambda\mu^N \psi_N(\lambda)\psi_N(\mu^{-1}) - \lambda^N \mu \psi_N(\lambda^{-1})\psi_N(\mu)}{\lambda - \mu} = \frac{\det A_{N+1}}{\det A_N} \sum_{n,m=0}^N \big(A_{N+1}^{-1}\big)_{m+1,n+1} \lambda^{N-n}\mu^m. \tag{15}$$

Comparison of this equation with $m = 0$ to (14) immediately leads to the identification of $(-1)^N \varphi_{N,\xi}(\lambda)$ with the Szegő polynomial $\psi_N(\lambda)$ of degree N. These polynomials satisfy recursion relations which for real symmetric coefficients take the form

$$\gamma_N = -\frac{1}{\delta_{N-1}} \oint \frac{\mathrm{d}\lambda}{2\pi \mathrm{i}} \psi_{N-1}(\lambda) \sum_{n=-\infty}^{\infty} c_n \lambda^n, \tag{16a}$$

$$\psi_N(\lambda) = \lambda \psi_{N-1}(\lambda) + \gamma_N \lambda^{N-1} \psi_{N-1}(\lambda^{-1}), \tag{16b}$$

$$\delta_N = \delta_{N-1}\left(1 - \gamma_N^2\right). \tag{16c}$$

The initial conditions are $\delta_0 = c_0$, $\psi_0(\lambda) = 1$. The numbers γ_N are known as the Schur or Verblunsky coefficients.

It can now be seen in an explicit if tedious calculation that the polynomials

$$\psi_N(\lambda) = (-1)^N \varphi_{N,\xi}(\lambda) = \sum_{n=0}^{N} \frac{(N-n+1)^{(\xi)}(n+1)^{(\xi-1)}}{(N+1)^{(\xi)}(1)^{(\xi-1)}} \lambda^{N-n} \tag{17a}$$

$$= \lambda^N {}_2F_1(-N, \xi; -N-\xi; \lambda^{-1}) \tag{17b}$$

(with coefficients and expansion given in (4) and (5)) indeed fulfil the Szegő recursion generated by the binomial coefficients $c_n = (-1)^n \binom{2\xi}{\xi-n}$. The recursion coefficients take the simple form

$$\gamma_N = \frac{\xi}{\xi + N}, \qquad \delta_N = \frac{N!(2\xi+1)^{(N)}}{[(\xi+1)^{(N)}]^2}. \tag{17c}$$

This completes the proof of (4) and also entails the validity of (2).

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